Streamlike Function: A New Concept in Flow Problems Formulation

A. Hamed* and S. Abdallah†
University of Cincinnati, Cincinnati, Ohio

Abstract

THE simultaneous solution of the nonhomogeneous equations representing the continuity and vorticity is frequently required in flow computations. Previous formulations resulted in two uncoupled second-order differential equations. A new approach is presented here, which is applicable in generalized two-dimensional and axisymmetric domains. It is based on the definition of a streamlike function which transforms the two first-order equations to a single second-order equation with Dirichlet boundary conditions, when the normal velocities are specified over the boundaries.

Contents

The continuity and vorticity equations can generally be written in the following form:

$$\frac{\partial}{\partial x}(x^j u) + \frac{\partial}{\partial y}(x^j v) = x^j S(x,y) \tag{1}$$

$$\frac{\partial}{\partial y}(u) - \frac{\partial}{\partial x}(v) = -\omega(x,y) \tag{2}$$

where (u,v) are the velocity components in (x,y) directions; S is the source/sink term; ω is the vorticity component in the direction normal to the x,y plane; and j has the values of zero and one for the two-dimensional and axisymmetric fields, respectively.

In the present study, a new approach for the solution of Eqs. (1) and (2) is presented. It is based on the definition of a dependent variable, which transforms the two equations to a single second-order equation. The new variable will be referred to as a "streamlike function" and is defined such that Eq. (1) is identically satisfied. The velocity components u and v are expressed in terms of the streamlike function, χ_1, and the source term, S, as follows:

$$u = \frac{1}{x^j}\frac{\partial \chi_1}{\partial y} + \frac{1}{x^j}\int_{x_r}^{x} x^j S(x,y)\,\mathrm{d}x \tag{3}$$

and

$$v = -\frac{1}{x^j}\frac{\partial \chi_1}{\partial x} \tag{4}$$

where x_r is the x coordinate at some reference line.

When Eqs. (3) and (4) are substituted into Eq. (2), we obtain

$$\frac{\partial^2 \chi_1}{\partial x^2} + \frac{\partial^2 \chi_1}{\partial y^2} - \frac{j}{x}\frac{\partial \chi_1}{\partial x} = -\sigma_1(x,y) \tag{5a}$$

where

$$\sigma_1(x,y) = x^j \omega(x,y) + \frac{\partial}{\partial y}\int_{x_r}^{x} x^j S(x,y)\,\mathrm{d}x \tag{5b}$$

The nonhomogeneous term, σ_1, in Eq. (5b) is dependent on the source term, S; the vorticity, ω; and on the choice of the reference line.

Alternatively, another streamlike function, χ_2, could also be defined to accomplish the same task. The integral term in this case can be introduced in the velocity component, v, as follows:

$$u = \frac{1}{x^j}\frac{\partial \chi_2}{\partial y} \tag{6}$$

and

$$v = -\frac{1}{x^j}\frac{\partial \chi_2}{\partial x} + \frac{1}{x^j}\int_{y_r}^{y} x^j S(x,y)\,\mathrm{d}y \tag{7}$$

In this case, y_r is the y coordinate at a reference line.

The streamlike function χ_2 also satisfies Eq. (1) identically. When Eqs. (6) and (7) are substituted into Eq. (2), one obtains

$$\frac{\partial^2 \chi_2}{\partial x^2} + \frac{\partial^2 \chi_2}{\partial y^2} - \frac{j}{x}\frac{\partial \chi_2}{\partial x} = -\sigma_2(x,y) \tag{8a}$$

where

$$\sigma_2(x,y) = x^j \omega(x,y) - x^j \frac{\partial}{\partial x}\int_{y_r}^{y} S(x,y)\,\mathrm{d}y \tag{8b}$$

The boundary conditions for χ_1 or χ_2 are dependent on the type of boundary conditions imposed on the flow velocities. Since our interest in the problem and, consequently, the examples presented are related to internal flows in turbomachinery passages, only the corresponding boundary conditions are discussed here. The following Dirichlet boundary conditions apply for the streamlike function when the normal velocity is specified over the boundaries:

$$\chi_1 = \int x^j V_n \,\mathrm{d}l - \int\int_{x_r}^{x} x^j S(x,y)\,\mathrm{d}x\mathrm{d}y + \text{constant} \tag{9}$$

and in terms of the streamlike function χ_2 as

$$\chi_2 = \int x^j V_n \,\mathrm{d}l + \int\int_{y_r}^{y} x^j S(x,y)\,\mathrm{d}y\mathrm{d}x + \text{constant} \tag{10}$$

where V_n is the specified normal velocity and $\mathrm{d}l$ is the incremental contour length. The boundary conditions for the

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*Associate Professor, Dept. of Aerospace Engineering and Applied Mechanics. Member AIAA.

†Graduate Research Assistant, Dept. of Aerospace Engineering and Applied Mechanics.

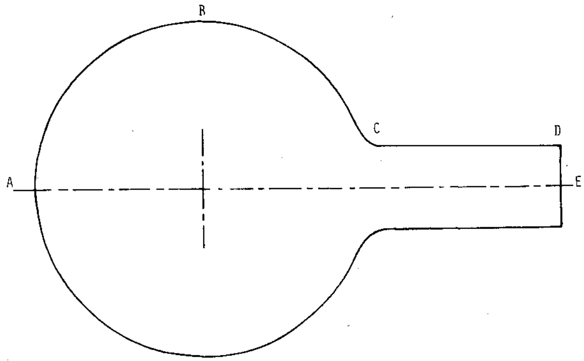

Fig. 1 Scroll cross-sectional geometry.

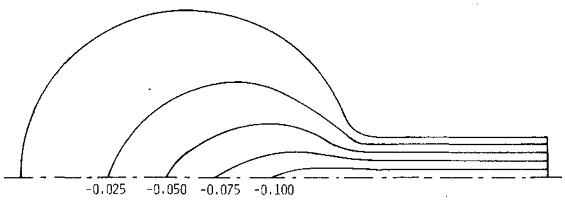

Fig. 2 Streamlike function χ_2 contours y_r on the upper side.

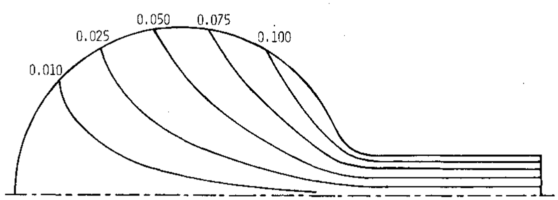

Fig. 3 Streamlike function χ_2 contours y_r on the lower side.

streamlike functions are defined within an arbitrary constant, as is the solution itself. Further details on the boundary conditions for other cases can be found in the original paper.

It is important to stress here that in the present analysis, a single second-order equation in one of the streamlike functions, χ_1 or χ_2, is needed in the solution. The corresponding boundary values, as well as the nonhomogeneous term in the resulting Poisson equation, are dependent on the choice of a particular formulation and of the reference line. The computed values of the flow velocities u and v are, however, independent of the type of streamlike function used in the analysis and of the reference line location. The flexibility in the present approach in terms of these available choices can be used advantageously, depending on the particular problem under consideration and on the numerical method used in the solution of the resulting equations.

One example is presented to illustrate the choices available in the application of the new formulation. It corresponds to the inviscid flow in the cross-sectional planes of the turbine scroll shown schematically in Fig. 1. In this case the source term is dependent on the through-flow velocity variations, and the normal flow velocities are specified over all of the solution domain boundaries.[1] This includes the uniform discharge velocity over the scroll exit, *DE*; zero normal velocities over the axis of symmetry, *AE*; and the irregular scroll boundary, *ABCD*.

The resulting streamlike function χ_2 contours are shown in Figs. 2 and 3 in one-half of the symmetric scroll for two different choices for the location of the reference line above and below the solution domain respectively. As can be seen from these figures, the streamlike function remains constant over the impermeable solid boundaries facing the reference line. The particular formulation can therefore be chosen such that the reference line can be placed on the side of the boundary with the most irregular configurations, which would be more convenient in the numerical solution.

Although this example does not represent the most generalized case, since the vorticity ω in the rotationality equation is equal to zero everywhere in the flowfield, it was intentionally chosen to illustrate a different application of the streamlike function. This same example has been formulated prior to this work in terms of a potential function[1] which resulted in Poisson equation with Neumann boundary conditions. The streamlike function has been used in effect to transform a Poisson-Neumann problem to a Poisson-Dirichlet problem. At present, the fast and direct numerical methods for solving the Poisson equation are applicable to problems with Dirichlet boundary conditions over arbitrary domains but not to problems with Neumann boundary conditions over irregular boundaries. The applicability of these methods could be extended through this tranformation to the second class of problems.

Acknowledgments

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Reference

[1] Hamed, A., Abdallah, S., and Tabakoff, W., "Flow Study in the Cross Sectional Planes of a Turbine Scroll," AIAA Paper 77-714.